Problem 4.65

If you combine *three* spin-1/2 particles, you can get a total spin of 3/2 or 1/2 (and the latter can be achieved in two distinct ways). Construct the quadruplet and the two doublets, using the notation of Equations 4.175 and 4.176:

$$\left.\begin{cases} \left|\frac{3}{2}\,\frac{3}{2}\right\rangle & = \;?? \\ \left|\frac{3}{2}\,\frac{1}{2}\right\rangle & = \;?? \\ \left|\frac{3}{2}\,\frac{-1}{2}\right\rangle & = \;?? \\ \left|\frac{3}{2}\,\frac{-3}{2}\right\rangle & = \;?? \end{cases}\right\} \quad s = \frac{3}{2} \text{ (quadruplet)}$$

$$\left.\begin{cases} \left|\frac{1}{2}\,\frac{1}{2}\right\rangle_1 & = \;?? \\ \left|\frac{1}{2}\,\frac{-1}{2}\right\rangle_1 & = \;?? \end{cases}\right\} \quad s = \frac{1}{2} \text{ (doublet 1)}$$

$$\left.\begin{cases} \left|\frac{1}{2}\,\frac{1}{2}\right\rangle_2 & = \;?? \\ \left|\frac{1}{2}\,\frac{-1}{2}\right\rangle_2 & = \;?? \end{cases}\right\} \quad s = \frac{1}{2} \text{ (doublet 2)}$$

Hint: The first one is easy: $\left|\frac{3}{2}\,\frac{3}{2}\right\rangle = |\uparrow\uparrow\uparrow\rangle$; apply the lowering operator to get the other states in the quadruplet. For the doublets you might start with the first two in the singlet state, and tack on the third:

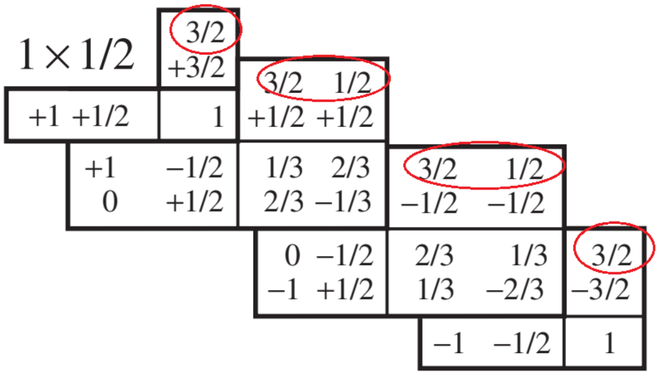
$$\left|\frac{1}{2}\,\frac{1}{2}\right\rangle_1 = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\,|\uparrow\rangle\,.$$

Take it from there (make sure $\left|\frac{1}{2}\,\frac{1}{2}\right\rangle_2$ is orthogonal to $\left|\frac{1}{2}\,\frac{1}{2}\right\rangle_1$ and to $\left|\frac{3}{2}\,\frac{1}{2}\right\rangle$). *Note:* the two doublets are not uniquely determined—any linear combination of them would still carry spin 1/2. The point is to construct two *independent* doublets.

Solution

Mr. Griffiths demonstrated in Example 4.5 on page 177 that two spin-1/2 particles can carry a total spin of 0 or 1. If the total spin is 0, then adding a third spin-1/2 particle will make the total spin 1/2. If the total spin is 1, then adding a third spin-1/2 particle will make the total spin either 3/2 or 1/2, depending if it's spin up or spin down.

One can see this in the $1 \times 1/2$ Clebsch–Gordon table.



If the total spin of the three spin-1/2 particles is $s = 3/2$, then $m_s = -3/2, -1/2, 1/2,$ or $3/2$, and there's a quadruplet of spin states. Each spin state is denoted by $|s\ m_s\rangle$.

$$\begin{cases} \left|\frac{3}{2}\ \frac{3}{2}\right\rangle \\ \left|\frac{3}{2}\ \frac{1}{2}\right\rangle \\ \left|\frac{3}{2}\ \frac{-1}{2}\right\rangle \\ \left|\frac{3}{2}\ \frac{-3}{2}\right\rangle \end{cases}$$

If the total spin of the three spin-1/2 particles is $s = 1/2$, then $m_s = -1/2$ or $m_s = 1/2$. There are two ways to achieve $s = 1/2$, so there are two doublets.

$$\begin{cases} \left|\frac{1}{2}\ \frac{1}{2}\right\rangle_1 \\ \left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_1 \end{cases}$$

$$\begin{cases} \left|\frac{1}{2}\ \frac{1}{2}\right\rangle_2 \\ \left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_2 \end{cases}$$

Let the first particle be in the state $|s_1\ m_1\rangle$, let the second particle be in the state $|s_2\ m_2\rangle$, and let the third particle be in the state $|s_3\ m_3\rangle$. The lowering operator is defined in Equation 4.136 on page 166.

$$S_-|s\ m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s-1)}\,|s\ (m_s-1)\rangle$$

Begin with the quadruplet associated with $s = 3/2$. The first spin state is given in the hint and represents the combination of three spin-1/2 particles with spin up:

$$\begin{aligned}\left|\frac{3}{2}\,\frac{3}{2}\right\rangle &= |\uparrow\uparrow\uparrow\rangle \\ &= \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle.\end{aligned}$$

Apply the lowering operator to both sides to get $\left|\frac{3}{2}\,\frac{1}{2}\right\rangle$.

$$S_-\left|\frac{3}{2}\,\frac{3}{2}\right\rangle = S_-\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle$$

$$\hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)-\frac{3}{2}\left(\frac{3}{2}-1\right)}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle = \left[S_-^{(1)}+S_-^{(2)}+S_-^{(3)}\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle$$

$$\begin{aligned}\hbar\sqrt{3}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle &= \left[S_-^{(1)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\ &\quad + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left[S_-^{(2)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\ &\quad + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left[S_-^{(3)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right] \\ &= \left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\ &\quad + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\ &\quad + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right] \\ &= \hbar\left(\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right. \\ &\quad + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\ &\quad \left. + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right) \\ &= \hbar(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)\end{aligned}$$

Therefore, dividing both sides by $\hbar\sqrt{3}$,

$$\boxed{\left|\frac{3}{2}\,\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(\left|\downarrow\uparrow\uparrow\right\rangle + \left|\uparrow\downarrow\uparrow\right\rangle + \left|\uparrow\uparrow\downarrow\right\rangle).}$$

Apply the lowering operator to both sides again to get $\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle$.

$$S_-\left|\frac{3}{2}\,\frac{1}{2}\right\rangle = S_-\left[\frac{1}{\sqrt{3}}(\left|\downarrow\uparrow\uparrow\right\rangle + \left|\uparrow\downarrow\uparrow\right\rangle + \left|\uparrow\uparrow\downarrow\right\rangle)\right]$$

$$\hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle = \frac{1}{\sqrt{3}}(S_-\left|\downarrow\uparrow\uparrow\right\rangle + S_-\left|\uparrow\downarrow\uparrow\right\rangle + S_-\left|\uparrow\uparrow\downarrow\right\rangle)$$

$$\begin{aligned}
\hbar\sqrt{4}\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle = \frac{1}{\sqrt{3}}\Bigg(&S_-\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\
&+ S_-\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\
&+ S_-\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\Bigg)
\end{aligned}$$

$$\begin{aligned}
2\hbar\sqrt{3}\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle &= \left[S_-^{(1)} + S_-^{(2)} + S_-^{(3)}\right]\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\
&\quad + \left[S_-^{(1)} + S_-^{(2)} + S_-^{(3)}\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\
&\quad + \left[S_-^{(1)} + S_-^{(2)} + S_-^{(3)}\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle \\
&= \left[S_-^{(1)}\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle + \left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left[S_-^{(2)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\
&\quad + \left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left[S_-^{(3)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right] + \left[S_-^{(1)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right]\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \\
&\quad + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left[S_-^{(2)}\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\left[S_-^{(3)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right] \\
&\quad + \left[S_-^{(1)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right]\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left[S_-^{(2)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\right]\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle \\
&\quad + \left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left[S_-^{(3)}\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right]
\end{aligned}$$

Evaluate the terms in square brackets.

$$\begin{aligned}
2\hbar\sqrt{3}\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle &= \left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)+\frac{1}{2}\left(-\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-3}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{1}{2}\right\rangle \\
&\quad + \left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right] + \left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle \\
&\quad + \left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)+\frac{1}{2}\left(-\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-3}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right] \\
&\quad + \left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle + \left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle \\
&\quad + \left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left[\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)+\frac{1}{2}\left(-\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-3}{2}\right\rangle\right] \\
&= \left[\hbar(0)\left|\frac{1}{2}\ \frac{-3}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left[\hbar(1)\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{1}{2}\right\rangle \\
&\quad + \left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left[\hbar(1)\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right] + \left[\hbar(1)\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle \\
&\quad + \left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left[\hbar(0)\left|\frac{1}{2}\ \frac{-3}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left[\hbar(1)\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right] \\
&\quad + \left[\hbar(1)\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle + \left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left[\hbar(1)\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right]\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle \\
&\quad + \left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left[\hbar(0)\left|\frac{1}{2}\ \frac{-3}{2}\right\rangle\right]
\end{aligned}$$

Simplify the right side.

$$2\hbar\sqrt{3}\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle = 2\hbar\left(\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle + \left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\right)$$

Therefore, dividing both sides by $2\hbar\sqrt{3}$,

$$\boxed{\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle = \frac{1}{\sqrt{3}}(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle).}$$

Apply the lowering operator to both sides again to get $\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle$.

$$S_-\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle = S_-\left[\frac{1}{\sqrt{3}}(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle)\right]$$

$$\hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)+\frac{1}{2}\left(-\frac{1}{2}-1\right)}\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle = \frac{1}{\sqrt{3}}(S_-|\downarrow\downarrow\uparrow\rangle + S_-|\downarrow\uparrow\downarrow\rangle + S_-|\uparrow\downarrow\downarrow\rangle)$$

$$\hbar\sqrt{3}\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(S_-|\downarrow\rangle|\downarrow\rangle|\uparrow\rangle + S_-|\downarrow\rangle|\uparrow\rangle|\downarrow\rangle + S_-|\uparrow\rangle|\downarrow\rangle|\downarrow\rangle\right)$$

$$\begin{aligned}3\hbar\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle &= \left[S_-^{(1)}+S_-^{(2)}+S_-^{(3)}\right]|\downarrow\rangle|\downarrow\rangle|\uparrow\rangle \\ &\quad + \left[S_-^{(1)}+S_-^{(2)}+S_-^{(3)}\right]|\downarrow\rangle|\uparrow\rangle|\downarrow\rangle \\ &\quad + \left[S_-^{(1)}+S_-^{(2)}+S_-^{(3)}\right]|\uparrow\rangle|\downarrow\rangle|\downarrow\rangle \\ &= \left[S_-^{(1)}|\downarrow\rangle\right]|\downarrow\rangle|\uparrow\rangle + |\downarrow\rangle\left[S_-^{(2)}|\downarrow\rangle\right]|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle\left[S_-^{(3)}|\uparrow\rangle\right] \\ &\quad + \left[S_-^{(1)}|\downarrow\rangle\right]|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle\left[S_-^{(2)}|\uparrow\rangle\right]|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\left[S_-^{(3)}|\downarrow\rangle\right] \\ &\quad + \left[S_-^{(1)}|\uparrow\rangle\right]|\downarrow\rangle|\downarrow\rangle + |\uparrow\rangle\left[S_-^{(2)}|\downarrow\rangle\right]|\downarrow\rangle + |\uparrow\rangle|\downarrow\rangle\left[S_-^{(3)}|\downarrow\rangle\right]\end{aligned}$$

Evaluate the terms in square brackets.

$$\begin{aligned}
3\hbar\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle &= (0)\left|\downarrow\right\rangle\left|\uparrow\right\rangle + \left|\downarrow\right\rangle(0)\left|\uparrow\right\rangle + \left|\downarrow\right\rangle\left|\downarrow\right\rangle[\hbar\left|\downarrow\right\rangle] \\
&\quad + (0)\left|\uparrow\right\rangle\left|\downarrow\right\rangle + \left|\downarrow\right\rangle[\hbar\left|\downarrow\right\rangle]\left|\downarrow\right\rangle + \left|\downarrow\right\rangle\left|\uparrow\right\rangle(0) \\
&\quad + [\hbar\left|\downarrow\right\rangle]\left|\downarrow\right\rangle\left|\downarrow\right\rangle + \left|\uparrow\right\rangle(0)\left|\downarrow\right\rangle + \left|\uparrow\right\rangle\left|\downarrow\right\rangle(0) \\
&= 3\hbar\left|\downarrow\right\rangle\left|\downarrow\right\rangle\left|\downarrow\right\rangle \\
&= 3\hbar\left|\downarrow\downarrow\downarrow\right\rangle
\end{aligned}$$

Therefore, dividing both sides by $3\hbar$,

$$\boxed{\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle = \left|\downarrow\downarrow\downarrow\right\rangle.}$$

Now on to the first doublet. One way for the three spin-1/2 particles to produce a total spin of 1/2 with $m_s = 1/2$ (spin up) is to have two particles with total spin 0 combined with a spin-1/2 particle with spin up. The formula for $|0\ 0\rangle$ is given in Equation 4.176 on page 177.

$$\begin{aligned}
\left|\frac{1}{2}\ \frac{1}{2}\right\rangle_1 &= |0\ 0\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle \\
&= \frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle)\left|\frac{1}{2}\ \frac{1}{2}\right\rangle \\
&= \frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle)\left|\uparrow\right\rangle
\end{aligned}$$

Therefore,

$$\boxed{\left|\frac{1}{2}\ \frac{1}{2}\right\rangle_1 = \frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\uparrow\right\rangle - \left|\downarrow\uparrow\uparrow\right\rangle).}$$

Apply the lowering operator to both sides to get $\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_1$.

$$\begin{aligned}
S_-\left|\frac{1}{2}\ \frac{1}{2}\right\rangle_1 &= S_-\left[\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\uparrow\right\rangle - \left|\downarrow\uparrow\uparrow\right\rangle)\right] \\
\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_1 &= \frac{1}{\sqrt{2}}(S_-\left|\uparrow\downarrow\uparrow\right\rangle - S_-\left|\downarrow\uparrow\uparrow\right\rangle) \\
\hbar(1)\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_1 &= \frac{1}{\sqrt{2}}\left\{\left[S_-^{(1)} + S_-^{(2)} + S_-^{(3)}\right]\left|\uparrow\downarrow\uparrow\right\rangle\right. \\
&\quad \left. - \left[S_-^{(1)} + S_-^{(2)} + S_-^{(3)}\right]\left|\downarrow\uparrow\uparrow\right\rangle\right\}
\end{aligned}$$

Multiply both sides by $\sqrt{2}$ and evaluate the right side.

$$\begin{aligned}
\hbar\sqrt{2}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_1 &= \left[S_-^{(1)} + S_-^{(2)} + S_-^{(3)}\right]|\uparrow\rangle\,|\downarrow\rangle\,|\uparrow\rangle \\
&\qquad - \left[S_-^{(1)} + S_-^{(2)} + S_-^{(3)}\right]|\downarrow\rangle\,|\uparrow\rangle\,|\uparrow\rangle \\
&= \left[S_-^{(1)}\,|\uparrow\rangle\right]|\downarrow\rangle\,|\uparrow\rangle + |\uparrow\rangle\left[S_-^{(2)}\,|\downarrow\rangle\right]|\uparrow\rangle + |\uparrow\rangle\,|\downarrow\rangle\left[S_-^{(3)}\,|\uparrow\rangle\right] \\
&\qquad - \left[S_-^{(1)}\,|\downarrow\rangle\right]|\uparrow\rangle\,|\uparrow\rangle - |\downarrow\rangle\left[S_-^{(2)}\,|\uparrow\rangle\right]|\uparrow\rangle - |\downarrow\rangle\,|\uparrow\rangle\left[S_-^{(3)}\,|\uparrow\rangle\right] \\
&= [\hbar\,|\downarrow\rangle]\,|\downarrow\rangle\,|\uparrow\rangle + |\uparrow\rangle\,(0)\,|\uparrow\rangle + |\uparrow\rangle\,|\downarrow\rangle\,[\hbar\,|\downarrow\rangle] \\
&\qquad - (0)\,|\uparrow\rangle\,|\uparrow\rangle - |\downarrow\rangle\,[\hbar\,|\downarrow\rangle]\,|\uparrow\rangle - |\downarrow\rangle\,|\uparrow\rangle\,[\hbar\,|\downarrow\rangle] \\
&= \hbar(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle)
\end{aligned}$$

Therefore, dividing both sides by $\hbar\sqrt{2}$,

$$\boxed{\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_1 = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle).}$$

The two formulas for $\left|\frac{3}{2}\,\frac{1}{2}\right\rangle$ and $\left|\frac{1}{2}\,\frac{1}{2}\right\rangle_1$ are as follows.

$$\begin{cases} \left|\dfrac{3}{2}\,\dfrac{1}{2}\right\rangle = \dfrac{1}{\sqrt{3}}\left|\downarrow\uparrow\uparrow\right\rangle + \dfrac{1}{\sqrt{3}}\left|\uparrow\downarrow\uparrow\right\rangle + \dfrac{1}{\sqrt{3}}\left|\uparrow\uparrow\downarrow\right\rangle \\[2ex] \left|\dfrac{1}{2}\,\dfrac{1}{2}\right\rangle_1 = -\dfrac{1}{\sqrt{2}}\left|\downarrow\uparrow\uparrow\right\rangle + \dfrac{1}{\sqrt{2}}\left|\uparrow\downarrow\uparrow\right\rangle + 0\left|\uparrow\uparrow\downarrow\right\rangle \end{cases}$$

Following the hint, obtain $\left|\frac{1}{2}\,\frac{1}{2}\right\rangle_2$ by requiring it to be orthogonal to both $\left|\frac{1}{2}\,\frac{1}{2}\right\rangle_1$ and $\left|\frac{3}{2}\,\frac{1}{2}\right\rangle$. Assume that it has the same form as these two but with unknown coefficients.

$$\left|\frac{1}{2}\,\frac{1}{2}\right\rangle_2 = A\left|\downarrow\uparrow\uparrow\right\rangle + B\left|\uparrow\downarrow\uparrow\right\rangle + C\left|\uparrow\uparrow\downarrow\right\rangle$$

Orthogonalization:
$$\begin{cases} \left\langle\dfrac{1}{2}\,\dfrac{1}{2}\right|_1 \cdot \left|\dfrac{1}{2}\,\dfrac{1}{2}\right\rangle_2 = 0 \\[2ex] \left\langle\dfrac{3}{2}\,\dfrac{1}{2}\right| \cdot \left|\dfrac{1}{2}\,\dfrac{1}{2}\right\rangle_2 = 0 \end{cases}$$

$$\begin{cases} 0 = \left(-\dfrac{1}{\sqrt{2}}\left\langle\downarrow\uparrow\uparrow\right| + \dfrac{1}{\sqrt{2}}\left\langle\uparrow\downarrow\uparrow\right|\right)\left(A\left|\downarrow\uparrow\uparrow\right\rangle + B\left|\uparrow\downarrow\uparrow\right\rangle + C\left|\uparrow\uparrow\downarrow\right\rangle\right) \\[2ex] 0 = \left(\dfrac{1}{\sqrt{3}}\left\langle\downarrow\uparrow\uparrow\right| + \dfrac{1}{\sqrt{3}}\left\langle\uparrow\downarrow\uparrow\right| + \dfrac{1}{\sqrt{3}}\left\langle\uparrow\uparrow\downarrow\right|\right)\left(A\left|\downarrow\uparrow\uparrow\right\rangle + B\left|\uparrow\downarrow\uparrow\right\rangle + C\left|\uparrow\uparrow\downarrow\right\rangle\right) \end{cases}$$

$$\begin{cases} 0 = -\dfrac{A}{\sqrt{2}}\left\langle\downarrow\uparrow\uparrow\,|\,\downarrow\uparrow\uparrow\right\rangle + \dfrac{B}{\sqrt{2}}\left\langle\uparrow\downarrow\uparrow\,|\,\uparrow\downarrow\uparrow\right\rangle \\[2ex] 0 = \dfrac{A}{\sqrt{3}}\left\langle\downarrow\uparrow\uparrow\,|\,\downarrow\uparrow\uparrow\right\rangle + \dfrac{B}{\sqrt{3}}\left\langle\uparrow\downarrow\uparrow\,|\,\uparrow\downarrow\uparrow\right\rangle + \dfrac{C}{\sqrt{3}}\left\langle\uparrow\uparrow\downarrow\,|\,\uparrow\uparrow\downarrow\right\rangle \end{cases}$$

$$\begin{cases} 0 = -\dfrac{A}{\sqrt{2}} + \dfrac{B}{\sqrt{2}} \\[2ex] 0 = \dfrac{A}{\sqrt{3}} + \dfrac{B}{\sqrt{3}} + \dfrac{C}{\sqrt{3}} \end{cases}$$

$$\begin{cases} 0 = -A + B \\[2ex] 0 = A + B + C \end{cases}$$

The third equation needed to determine A, B, and C comes from normalization:

$$|A|^2 + |B|^2 + |C|^2 = 1.$$

Plugging the first two into the third gives

$$B^2 + B^2 + 4B^2 = 1$$

$$6B^2 = 1$$

$$B = \pm\frac{1}{\sqrt{6}},$$

which means

$$A = \pm\frac{1}{\sqrt{6}} \quad \text{and} \quad C = \mp\frac{2}{\sqrt{6}}.$$

Therefore, choosing the signs on top,

$$\left|\frac{1}{2}\ \frac{1}{2}\right\rangle_2 = \frac{1}{\sqrt{6}}\left|\downarrow\uparrow\uparrow\right\rangle + \frac{1}{\sqrt{6}}\left|\uparrow\downarrow\uparrow\right\rangle - \frac{2}{\sqrt{6}}\left|\uparrow\uparrow\downarrow\right\rangle$$

$$\boxed{\left|\frac{1}{2}\ \frac{1}{2}\right\rangle_2 = \frac{1}{\sqrt{6}}(\left|\downarrow\uparrow\uparrow\right\rangle + \left|\uparrow\downarrow\uparrow\right\rangle - 2\left|\uparrow\uparrow\downarrow\right\rangle).}$$

Apply the lowering operator to both sides to get $\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_2$.

$$S_-\left|\frac{1}{2}\ \frac{1}{2}\right\rangle_2 = S_-\left[\frac{1}{\sqrt{6}}(\left|\downarrow\uparrow\uparrow\right\rangle + \left|\uparrow\downarrow\uparrow\right\rangle - 2\left|\uparrow\uparrow\downarrow\right\rangle)\right]$$

$$\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_2 = \frac{1}{\sqrt{6}}\left(S_-\left|\downarrow\uparrow\uparrow\right\rangle + S_-\left|\uparrow\downarrow\uparrow\right\rangle - 2S_-\left|\uparrow\uparrow\downarrow\right\rangle\right)$$

$$\begin{aligned}\hbar(1)\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_2 = \frac{1}{\sqrt{6}}\big(&S_-\left|\downarrow\right\rangle\left|\uparrow\right\rangle\left|\uparrow\right\rangle \\ &+ S_-\left|\uparrow\right\rangle\left|\downarrow\right\rangle\left|\uparrow\right\rangle \\ &- 2S_-\left|\uparrow\right\rangle\left|\uparrow\right\rangle\left|\downarrow\right\rangle\big)\end{aligned}$$

$$\begin{aligned}\hbar\sqrt{6}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_2 = &\left[S_-^{(1)} + S_-^{(2)} + S_-^{(3)}\right]\left|\downarrow\right\rangle\left|\uparrow\right\rangle\left|\uparrow\right\rangle \\ &+ \left[S_-^{(1)} + S_-^{(2)} + S_-^{(3)}\right]\left|\uparrow\right\rangle\left|\downarrow\right\rangle\left|\uparrow\right\rangle \\ &- 2\left[S_-^{(1)} + S_-^{(2)} + S_-^{(3)}\right]\left|\uparrow\right\rangle\left|\uparrow\right\rangle\left|\downarrow\right\rangle\end{aligned}$$

Evaluate the right side.

$$\begin{aligned}
\hbar\sqrt{6}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_2 &= \left[S_-^{(1)}|\downarrow\rangle\right]|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle\left[S_-^{(2)}|\uparrow\rangle\right]|\uparrow\rangle + |\downarrow\rangle|\uparrow\rangle\left[S_-^{(3)}|\uparrow\rangle\right] \\
&\quad + \left[S_-^{(1)}|\uparrow\rangle\right]|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle\left[S_-^{(2)}|\downarrow\rangle\right]|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle\left[S_-^{(3)}|\uparrow\rangle\right] \\
&\quad - 2\left[S_-^{(1)}|\uparrow\rangle\right]|\uparrow\rangle|\downarrow\rangle - 2|\uparrow\rangle\left[S_-^{(2)}|\uparrow\rangle\right]|\downarrow\rangle - 2|\uparrow\rangle|\uparrow\rangle\left[S_-^{(3)}|\downarrow\rangle\right] \\
&= (0)|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle[\hbar|\downarrow\rangle]|\uparrow\rangle + |\downarrow\rangle|\uparrow\rangle[\hbar|\downarrow\rangle] \\
&\quad + [\hbar|\downarrow\rangle]|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle(0)|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle[\hbar|\downarrow\rangle] \\
&\quad - 2[\hbar|\downarrow\rangle]|\uparrow\rangle|\downarrow\rangle - 2|\uparrow\rangle[\hbar|\downarrow\rangle]|\downarrow\rangle - 2|\uparrow\rangle|\uparrow\rangle(0) \\
&= \hbar(2|\downarrow\rangle|\downarrow\rangle|\uparrow\rangle - |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle - |\uparrow\rangle|\downarrow\rangle|\downarrow\rangle)
\end{aligned}$$

Therefore, dividing both sides by $\hbar\sqrt{6}$,

$$\boxed{\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_2 = \frac{1}{\sqrt{6}}(2|\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle).}$$

In summary,

$$\left\{\begin{array}{lcl}
\left|\frac{3}{2}\ \frac{3}{2}\right\rangle &=& |\uparrow\uparrow\uparrow\rangle \\
\left|\frac{3}{2}\ \frac{1}{2}\right\rangle &=& \frac{1}{\sqrt{3}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \\
\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle &=& \frac{1}{\sqrt{3}}(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle) \\
\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle &=& |\downarrow\downarrow\downarrow\rangle
\end{array}\right\} \qquad s = \frac{3}{2}\ \text{(quadruplet)}$$

$$\left\{\begin{array}{lcl}
\left|\frac{1}{2}\ \frac{1}{2}\right\rangle_1 &=& \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle) \\
\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_1 &=& \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle)
\end{array}\right\} \qquad s = \frac{1}{2}\ \text{(doublet 1)}$$

$$\left\{\begin{array}{lcl}
\left|\frac{1}{2}\ \frac{1}{2}\right\rangle_2 &=& \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle) \\
\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle_2 &=& \frac{1}{\sqrt{6}}(2|\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle)
\end{array}\right\} \qquad s = \frac{1}{2}\ \text{(doublet 2).}$$